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SOLUTIONS TO SYSTEMS

- **Solution:** one list of numbers that satisfy a system
- **Solution set:** the set of all lists that satisfy a system
  - Notation: { | } is the set where everything behind the | is the conditional
- **Inconsistent:** no solution
- **Consistent, Independent:** one, unique solution
  - Needs as many unique equations as unknowns to find a unique solution for all unknowns
- **Consistent, Dependent:** infinitely many solutions

SYSTEMS

- **Homogenous** when a system is in the form:

$$a_{11}x_1 + \ldots + a_{1n}x_n = 0$$
$$\ldots$$
$$a_m x_1 + \ldots + a_{mn}x_n = 0$$

MATRICES

- **Upper Triangular:** when the nonzero entries are above or along the diagonal
  - The product of two upper triangular matrices is an upper triangular matrix
- **Lower Triangular:** when the nonzero entries are below or along the diagonal
  - The product of two lower triangular matrices is a lower triangular matrix
- **Diagonal:** when the only nonzero entries are along the diagonal
  - A diagonal matrix is both upper and lower triangular

MATRIX MULTIPLICATION

- The commutative property does not apply except for matrix inverses and identities.
- The coefficient matrix * (general) vector matrix = linear combination matrix of the column vectors

MATRIX TRANSPOSE

- Rows and Columns swap

- $(A + B)^T = A^T + B^T$

MATRIX ROW REDUCTION

- Works regardless of presence of augmentation
- 3 Legal **Row Operations**
  - Row Swap R_a <-> R_b
  - Row Multiply $c * R_a$ -> R_a
  - Row Addition $R_a + c * R_b$ -> R_a
- **Echelon Form:** the leading (first non-zero) entry in every row is strictly to the right of the leading entry (pivot) above it; has staircase structure of 0s below leading entries.
  - When a leading entry is in the augmented section of the matrix, the system is inconsistent.
- **Reduced Row Echelon Form:** the leading entries of the echelon form are equal to 1 AND they are the only nonzero entries of the column.
- **Free Variables:** any real number
  - one in every column without a leading entry
  - number of free variables equals the dimension of the solution set

MATRIX INVERTIBILITY

- Invertible when A and B are square matrices where AB = I = BA; $A = B^{-1}$; $A^{-1} = B$
- Conditions:
  - columns are linearly independent
  - columns span R^n
  - RREF is the identity matrix
- When an invertible matrix is augmented with the identity and reduced to the identity, the augmented side becomes the inverse.

LINEAR TRANSFORMATION

- preserves addition $T(v_1) + T(v_2) = T(v_1 + v_2)$
- preserves scalar multiplication $T(c * v) = c * T(v)$
- Composition of Linear Transformations: if T and S are linear transformations which transform R^n -> R^m -> R^k respectively, then S * T * R^n -> R^k
  - If T and S are linear transformations, S * T is also a linear transformation

- **Image** (of a linear transformation): set of all outputs of the linear transformation
    - synonymous with column space
    - the span of the columns of the transformation matrix
    - Dimension of Im(T) is the number of leading entries in echelon form
- **Kernel:** the set of all vectors mapped to 0 by the transformation matrices
    - synonymous with null space
    - Dimension of Ker(T) is the number of columns without a leading entry in echelon form
- **Rank Nullity Theorem:** Dim(Im(T)) + Dim(Ker(T)) = the number of columns of T

VECTORS

- point in n-dimensional space
- **Linear Combination:** when a vector is a scalar multiple of the sum of the other vectors
- **Standard Basis Vectors:** (e_i) has a 1 in the ith row and a 0 everywhere else
    - every vector can be written as a linear combination of the standard basis vectors
- **Span:** all the linear combinations of a vector list
    - Notation: ϵ means in the span
    - When a system spans the R^n, the solution is consistent
    - The solution set is every vector in R^n
    - Has a leading entry in every row in echelon form (not augmented)
- **Linearly Independent:** when only the trivial (all coefficients = 0) is the solution to a homogeneous system.
    - When the solution set is linearly independent, the solution is unique (as long as the system is consistent)
    - None of the vectors in the set is a linear combination of the others
    - Has a leading entry in every column in echelon form (not augmented)
- **Basis:** the solution spans R^n and the system is linearly independent
    - When the non-augmented portion of a matrix isn't square, the solution cannot be a basis
    - Any vector can be written in R^n as a linear combination of the basis elements
- **Dimension:** number of elements in a basis of R^n

VECTOR SPACE

- a set where addition and scalar multiplication within the space stays within the space (and other axioms must be true, too)

- There must be a unique vector, f, with the property that for any other vector, g, such that f + g = g (additive identity / the "zero" vector)

- **Subspace:** a subset of the original vector space in which it is still closed under addition and scalar multiplication
    - Every vector subspace is the span of some elements in the original set of vectors
    - Every vector subspace is the kernel of some linear transformation
    - The image of a subset of vectors is a subspace of the vector space
- **Isomorphism:** when there exists a one to one mapping between two vector spaces (an element is reversible when S(T(v)) = v; T(S(w)) = w)
    - Notation: ≅ means two vector spaces are isomorphic
    - If V is a vector space and S is a basis for it, then V is isomorphic to R^n and every vector in V can be written as a linear combination of the vectors in S
    - preserves length of the basis

BIJECTIVITY

- **Injective:** one to one mapping of input to output
    - Ker(T) = 0
    - Check for independence = check for injectivity (has a leading entry in every column in echelon form)
- **Surjective:** Image of the transformation is R^m given T * R^n -> R^m
    - columns of the transformation matrix span R^m
    - has a leading entry in every row in echelon form
- **Bijective:** both injective and surjective
    - Square transformation matrices that are injective or subjective are bijective
    - Bijective functions have an inverse, which is also bijective
    - T R^m -> R^m

DETERMINANTS

- 1 x 1 matrix = only element in the matrix
- 2 x 2 matrix = ad - bc; the matrix being defined as: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- For a diagonal and upper/lower triangular matrices, the determinant is the product of all the diagonal entries
- det A = $\sum_{j=1}^{n} (-1)^{1+j} * a_{1j} * det\ A_{1j}$

- element * determinant of the matrix minus the row and column containing the element, alternating in positivity based on position moving rightward or downward
- when a square matrix's det != 0, the matrix is invertible
- Properties
  - det(AB) = detA * detB
  - det(kA) = k^n * detA where n is the dimension of A
  - det(A^{-1}) = 1/detA
- **Characteristic Polynomial** = det(A - ƛI)
  - When det(A - ƛI) = 0, there must be something in the kernel; Ax = ƛx where A acts like scalar multiplication and ƛ is a scalar multiple
  - The degree is equal to the dimension of A
- **Additive Multiplicity:** the number of times a factor in a polynomial is repeated
- **Geometric Multiplicity:** dimension for the eigenspace for ƛ

EIGENVALUES & EIGENVECTORS

- **Eigenvalue:** ƛ is an eigenvalue for A when Ax = ƛx for some nonzero x (acts a scalar)
  - Roots of the characteristic polynomial
  - The eigenvalues of a diagonal matrix are the diagonal entries
- **Eigenvector:** x is an eigenvector if x is nonzero and Ax = ƛx for some scalar ƛ
  - A vector from the basis solution using the row-reduced form of (A - ƛI) * x = 0
  - Eigenvectors are in the Kernel of (A- ƛI)
  - Eigenvalues of AB = Eigenvalues of BA given that A and B are square matrices
  - The eigenvectors for a diagonal matrix are the standard basis vectors
  - There exists at least one set of eigenvalues and eigenvectors for an invertible matrix

DIAGONALIZATION

- A = PDP^{-1} where D is a diagonal matrix
  - There must exist a number of linearly independent eigenvectors which equal the dimension of A
  - P is a matrix of eigenvectors and D is a diagonal matrix of eigenvalues where the ith eigenvector is in the ith column of P and the ith eigenvalue is in the ith diagonal entry of D
  - A is diagonalizable when the geometric multiplicity = algebraic multiplicity for every eigenvalue

- **Fundamental Theorem of Algebra:** the number of complex roots that exist in a polynomial is the highest degree of that polynomial

EIGENSPACES

- The set of vectors in V which are eigenvectors for a certain eigenvalue ƛ where T is a linear transformation from V -> V and T(v) = ƛv
- Are always vector subspaces

COMPLEX NUMBERS

- If a + bi != 0, a + bi is invertible
- 2-dimensional vector space
  - Basis: { 1, i }
- Multiplication
  - i: rotation
  - real number: scaling
- a + bi = r * $e^{i * \Theta}$ where r = $\sqrt{a^2 + b^2}$ and Θ = arctan(b/a)

DOT PRODUCTS

- Dot (Inner) Product of two vectors v and w in R^n = v^T * w.
- **Properties**
  - communicative, associative, distributive
  - The dot product of u and u >= 0, the dot product of u and u = 0 exactly when u = 0.
- **Length (norm):** square root of the dot product of a vector by itself
  - ||c * v|| = |c| * ||v||
  - **Unit Vector:** a vector u such that ||u|| = 1
- **Direction:** two vectors v and u are in the same direction if v = c * u for some scalar c
- **Distance:** the distance between two vectors u and v = ||u - v||
- **Angle:** | u dot v | = ||u|| * ||v|| * cos Θ

ORTHOGONALITY

- Two vectors u and v are orthogonal (⊥) if the dot product of u and v = 0
  - u and v are orthogonal if and only if d(u, v) = d(u, -v)
  - u and v are orthogonal if and only if $||u + v||^2 = ||u||^2 + ||v||^2$

- **Orthocomplement:** all vectors in v in R^n such that $v \perp w$ for every w in W given that W is a subspace of R^n
    - **Notation:** $W^\perp$ (W perp)
    - Must check the subspace conditions for the orthocomplement of W
    - $(W^\perp)^\perp = W$
    - $(\text{Im } A)^\perp = \text{Ker } A^T$
- **Orthogonal Sets:** a set of vectors $\{u_1, \dots, u_p\}$ in R^n such that u_i != u_j where i != j.
    - Orthogonal sets are linearly independent and a basis for the subspace spanned by that set of vectors
- **Orthogonal Basis:** a basis that is also an orthogonal set in a subspace for W of R^n
- **Orthogonal Projection:** decomposition of a vector (y) into the parallel (y-hat) and perpendicular (z) components to another vector (u)
    - y-hat = projection map $P_u(y)$ = ((y dot u) / (u dot u)) * u
    - z = y - $P_u(y)$
- **Projection Matrices**
    - Properties: $P^2 = P$; $P = P^T$
- **Orthonormal:** a set of vectors $\{u_1, \dots, u_n\}$ if u_i dot u_j = 0 if i != j and u_i dot u_i = 0 for each i
- **Orthogonal (Unitary) Matrices:** A matrix U is orthogonal (unitary) if $U^T * U = U * U^T = I$
    - U is orthogonal if and only if the columns of U form an orthonormal basis for R^n
    - When U is orthogonal, U*x dot U*y = x dot y
    - U preserves length and angle, providing numerical stability (preserves dot product)
- **Orthogonal Projection:** a square matrix P with the property that $P^T = P$ and $P^2 = P$.
    - If Im(P) = W, P is an orthogonal projection onto W
    - Given a subspace W of R^n, a vector v in R^n can be written as v = w + z where w is a vector in W and z is a vector in $W^\perp$. w = $P_W(v)$
    - $\Sigma^p_{j=1} (e_i \text{ dot } w_j)\, w_j$ where i is the ith column in the projection matrix
    - For each w in W $P_W(w) = w$
    - Ker $P_W = W^\perp$
    - If w in W and v in R^n, v dot w = $P_W(v)$ dot w

LEAST SQUARES

- If A is not a square matrix, a least squares solution to the system A*x = b is a vector x with the property that || b - A * x-hat || <= || b - Ax || for any x in R^m
    - input that is closest to the output (for an inconsistent)

    - perpendicular to the image of A
    - $A^T * A * x = A^T * b$
    - There is exactly one least-square solution to a system when Ker A = { 0 }
    - $A^T * A$ is invertible if and only if Ker A = { 0 }

CHANGE OF BASIS

- Not on Final: Lecture 7, 12

GRAM-SCHMIDT

- Not on Final: Lecture 17