Differential Equation: equations containing a function and its derivative; solution is a function

Slope Fields:

- plotted with tangent lines at each incremental point
- follow trajectory of slope lines to determine behavior of solutions; never cross slope lines
- one initial condition is associated with one integral curve

Classification of Differentials:

- Ordinary: derivatives are only with respect to 1 variable
- Partial: contains partial derivatives dealing with more than 1 variable in its differential
- Order: highest derivative degree present in the equation
- Linear: equation cannot contain products of linear combinations of the function or its derivatives (only each individual component separate)

First ODE: (Linear) Integrating Factors

- Standard Form: $y' + p(t) * y = g(t)$
- General Method of Integrating Factors:
  - convert DE to standard form
  - multiply all terms by integrating factor $M(t)$
  - solve for $M(t)$ such that $M'(t) = M(t) * p(t)$
  - undo the product rule to find that $M(t) * y(t) = \int M(t) * g(t)\, dt$
  - solve for the explicit solution, $y(t)$

First ODE: Separation of Variables

- Separable if equations can be written in the form $M(x)\, dx + N(y)\, dy = 0$ [alternatively, $dy/dx = f(x) * g(y)$]
- General Method of Separating Variables:
  - separate variables onto different sides of the equation
  - integrate both sides to find the implicit solution
  - solve for function to determine the explicit solution
  - determine constant of integration using initial condition

First ODE: Existence and Uniqueness Theorem

- Linear Equations: If the functions p(t) and g(t) are on continuous on an open interval a < t < b containing the point t = t0, then there exists a unique function u(t) that satisfies the differential equation y' + p(t) * y = g(t) for each t in (a, b) which also satisfies the initial condition y(t0) = y0.
- Nonlinear Equations: If functions f and df/dy are continuous in some rectangle a < t < b and c < y < d containing the point (t0, y0). Then, in some interval t0 - h < t < t0 + h contained in a < t < b, there is a unique solution y = u(t) to the initial value problem y' = f(t, y); y(t0) = y0.

Properties of Autonomous Equations

- Autonomous Equations: equation where the derivative is equal to a function purely in the same variable (equations are in the form dy/dt = f(y).)
- Critical Points (Equilibrium Solutions): dy/dt = 0; y0 such that f(y0) = 0
    - Stable Critical Points: phase line directions both point toward this critical point
    - Semistable Critical Points: one phase line directions point toward this critical point
    - Unstable Critical Point: phase line directions both point away from this critical point
- Phase Line: line that indicates critical points and signs of the slopes of the intervals separated by critical points

Wronskian Determinant

- W[a, b] = $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$ (determinant)

Second ODE (Linear): Constant-Coefficient Homogeneous DE

- Standard Form: y'' + p(t) * y' + g(t) * y = q(t)
- Homogeneous Equations: when q(t) = 0
- General Method for Solving:
    - Guess the general solution $y = e^{r * t}$
    - Substitute y, y', and y'' to find the second degree characteristic polynomial with respect to r
    - Solve for the the roots of r
    - If the roots are real and unique, the general solution will be a linear combination of all possible general solutions ($y = c_1 * e^{r1 * t} + c_2 * e^{r2 * t}$)
    - If the roots are complex and unique, the general solution will be $e^a * (c_1 * \cos(b*t) + c_2 * \sin(b* t))$
    - If the roots are repeated, the general solution will be $c_1 * e^{r * t} + c_2 * t * e^{r * t}$

Second ODE (Linear, Non-Homogeneous): Undetermined Coefficients Method

- General Method for Solving:
  - Find the general solution of the associated homogeneous equation
  - Find any particular solution to the nonhomogeneous equation
  - Sum the general and particular solution to find the general solution of the nonhomogeneous equation
- General Method for determining the Undetermined Coefficients
  - Guess a particular solution and substitute it into the nonhomogeneous equation
    - If q(t) is a polynomial of degree n, guess a polynomial of the same degree: $y = At^n + Bt^{n-1} + \ldots Q$
    - If q(t) is an exponential function e^{rt}, guess a multiple of that function $y = Ae^{rt}$
    - If q(t) is a trigonomic function, guess a linear combination of the trig functions: y = Acos(rt) + Bsin(rt)
    - If q(t) can be expressed as a linear combination of the above, guess the sum of those components.
    - If q(t) is a product of the above functions, guess a product of those components.
    - If there exists duplications from the associated homogeneous equation, multiply by a term t^s such that there would not be duplication.
  - Determine the undetermined coefficients, if plausible, to find a particular solution that works

Second ODE (Particular Solutions): Variation of Parameters Method

- Given the associated homogeneous solution y = c1 * y1 + c2 * y2, one particular solution to the nonhomogeneous equation y'' + p(t) * y' + g(t) * y = q(t) is y = u1 * y1 + u2 * y2 where p(t), g(t), and q(t) are continuous on an open interval.
  - $u_1 = -\int \frac{y2}{y2' * y1 - y2 * y1'} * q(t)\,dt$
  - $u_2 = \int \frac{y1}{y2' * y1 - y2 * y1'} * q(t)\,dt$

Higher ODE (Linear)

- Given an nth degree DE, there needs to be n initial conditions to determine a unique solution.
- General solution for a homogeneous DE: y = c1 * y1 + ... + cn * yn
  - Coefficients can be determined by using a system of linear equations using provided initial conditions and the respective derivatives of the homogenous DE provided the determinant of the coefficient matrix is not zero
- **Fundamental set of solutions:** the set of solutions y1, ... , yn such that the general solution equates to a linear combination of the elements in this set
- **Linear Independence:** $\sum_{i=1}^{n} c_i * y_i = 0$ where c_i must be zero.
- General solution for non-homogeneous DE: general solution of associated homogeneous equation + one particular solution
  - In the event of a repeated root, multiply by t^n for the smallest n that makes the terms unique
- **Variation of Parameters**, general equation:
  - $u_n = \int \frac{W_n(t)}{W(t)} * q(t)\, dt$

Systems of Equations: Existence & Uniqueness Theorem

- For the system x'(t) = A(t) * x(t) + g(t) where x and g(t) is an n x 1 vector and A(t) is an n x n transformation matrix, if A(t) and g(t) are defined and continuous on the interval t:[a,b] then there must exist a solution on that interval.
- If $x_1(t)$, ..., $x_n(t)$ are linearly independent solutions to the system for each point on the interval t:[a,b], then each solution to the system can be expressed as a unique linear combination of $x_1(t)$, ..., $x_n(t)$.
- Each point on the interval t:[a, b] is said to generate a general solution to be a **fundamental set of solutions.**
- Every solution to the system can be represented as a linear combination of the fundamental set of solutions.

Phase Planes and Phase Portrait

- **Phase Plane:** graph the value of x with respect to t (can be 3D)
- **Phase Portrait:** projection of the phase plane onto the xy plane; graph of critical points and eigenvector lines and direction toward/away critical points (the origin).
    - When eigenvalues of the associated eigenvector is negative, the graph approaches the critical point as t ->∞; when the eigenvalue is positive, the graph goes away the critical point as t->∞.
- **Asymptotically Stable (Sink) Node (Local Max):** all eigenvectors approach the critical point
- **Unstable (Source) Node (Local Min):** all eigenvectors go away from the critical point
- **Star Point:** when eigenvalues are equal such that all trajectories become a line toward/away from the origin
- **Improper Node:** for a system when there exists generalized eigenvectors
- **Saddle Point:** one eigenvector approaches, while one goes away from the critical point

Systems of Equations: General Method of Solving

- For the homogenous system x' = A*x, guess x = v * $e^{\lambda t}$. Then, λ*v = A*v; thus, v must be an eigenvector with associated eigenvalue λ.
- Determine the characteristic polynomial by taking the determinant of A - λI and equating it to 0; solve for the eigenvalues.
- Determine the associated eigenvector(s), v, such that (A - λI) * v = 0.
    - For complex eigenvalues, only find the associated eigenvector for the root a + bi.
    - For repeated eigenvalues for a rank n system, if there are not n linearly independent eigenvectors, choose a generalized eigenvector in the form $e^{\lambda t}$ * (t * $v_{associated}$ + w) where (A - λI) * w = v.
        - Repeat this process a number of times equal to the algebraic multiplicity of the eigenvalue.
- Determine the fundamental set of solutions, a linear combination of all the $e^{\lambda t}$ * $v_{associated}$ pairs.
    - For complex eigenvalues and eigenvectors, use the fact that $e^{(a+bi)t}$ = e^{at} * (cos(b*t) + i * sin(b*t)) to determine the real and imaginary parts of the general solution. The fundamental set of solutions is a linear combination of the real and imaginary part.
    - For complex eigenvalues and eigenvectors, the fundamental set of solutions can be written as x = T * R * C * e^{at} where T is a transformation matrix of constants, R is the rotation matrix, and C is a vector of constants.
    - For complex eigenvalues and eigenvectors, when a = 0, the phase portrait is a circle (when T is a scalar of the identity) or an ellipse (when T isn't a scalar of the identity). The phase will be either a helix (**spiral point**), circle (**center**), or ellipse (**center**). Plug in any point to determine the orientation.

Systems of Equations: Diagonalizable Matrices and Jordan Canonical Form

- A matrix is diagonalizable if its determinant is not 0.
- For an n x n transformation matrix A(t) with n linearly independent eigenvectors, the $A = TDT^{-1}$ where T is the eigenvectors in an order which matches the column number of the diagonalized matrix D where the ith diagonal element is the associated eigenvalue of the ith column of T.
- When there are not n linearly independent eigenvectors, an almost diagonalized form, **Jordan Canonical**, can be achieved where there exists 1s above the diagonal where the general eigenvectors are dependent on the previous eigenvector.